Assignment 4: Surface Integrals: Solutions

1. (a) Calculating:

$$\text{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial y} = 2xz + xy$$

and

$$\begin{aligned} \text{curl}\mathbf{F} &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} \\ &= (xz - y\sin yz)\mathbf{i} - (yz - x^2)\mathbf{j} + (0-0)\mathbf{k} \\ &= (xz - y\sin yz)\mathbf{i} + (x^2 - yz)\mathbf{j} \end{aligned}$$

Since curl$\mathbf{F} \neq 0$, the vector field is not conservative.

(b) Calculating:

$$\text{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial y} = 6xz + ye^x$$

and

$$\begin{aligned} \text{curl}\mathbf{F} &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} \\ &= (0-0)\mathbf{i} - (3x^2 - 3x^2)\mathbf{j} + (e^x - e^x)\mathbf{k} \\ &= \mathbf{0} \end{aligned}$$

Since curl$\mathbf{F} = 0$, the vector field is conservative.

2. First, we need to calculate the vectors r_u, r_v at the parameter $u = 0, v = 1$. Taking partial derivatives gives

$$r_u =< v, 0, e^u + ue^u > \text{ and } r_v =< u, 2v, 0 >$$

Thus, at $u = 0, v = 1$, we get $r_u =< 1, 0, 1 >$ and $r_v =< 0, 2, 0 >$. The cross product of these two vectors is $-2\mathbf{i} + 2\mathbf{k}$. Thus gives a normal vector to the plane.

To find a point in the plane, we plug in $u = 0, v = 1$ into the parametric equation, giving the point $(0, 1, 0)$. Thus an equation of the plane is

$$-2(x-0) + 0(y-1) + 2(z-0) = 0$$

or, when simplified, $z = x$.

3. A parametrization of the surface is given by $< z-y, y, z >$ where (y, z) are in D, where D is the circle of radius 1 centred at the origin. We can then calculate

$$r_y =< -1, 1, 0 > \text{ and } r_z =< 1, 0, 1 >$$

The cross product $r_y \times r_z$ is $\mathbf{i}+\mathbf{j}-\mathbf{k}$. This has magnitude $\sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$. The surface area is then given by

$$\begin{aligned} & \int_D |r_y \times r_z| \, dA \\ = & \int_D \sqrt{3} \, dA \\ = & \sqrt{3} \int_D 1 \, dA \\ = & \sqrt{3}\pi \end{aligned}$$

Since the integral of 1 in D is the area of D, which is π, since D is a circle of radius 1.

4. The surface has parametrization $< y^2, y, z >$, where $1 \leq y \leq 4$ and $0 \leq z \leq 5$. We can then calculate

$$r_y =< 2y, 1, 0 > \text{ and } r_z =< 0, 0, 1 >$$

The cross product $r_y \times r_z$ is $\mathbf{i}-2y\mathbf{j}$. This has magnitude $\sqrt{1^2 + (-2y)^2 + 0^2} = \sqrt{4y^2 + 1}$. Thus the surface integral is

$$\begin{aligned} & \int_S \frac{xz}{y} \, dS \\ = & \int_D \frac{y^2 z}{y} \sqrt{4y^2 + 1} \, dA \\ = & \int_1^4 \int_0^5 yz\sqrt{4y^2 + 1} \, dz \, dy \\ = & \int_1^4 y\sqrt{4y^2 + 1} \left(\frac{z^2}{2} \right)_0^5 dy \\ = & \frac{25}{2} \int_1^4 y\sqrt{4y^2 + 1} \, dy \\ = & \frac{25}{24} (4y^2 + 1)^{3/2}|_1^4 \\ = & \frac{25}{24} (65^{3/2} - 5^{3/2}) \end{aligned}$$

5. We could solve this question using the usual parametrization of the sphere to get $r_\phi \times r_\theta$, but there is an easier way. A unit normal to the sphere $x^2 + y^2 + z^2 = a^2$ can be easily visualized, and is given by $\mathbf{n} = \frac{1}{a}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ (see example 6 on page 1090). Thus

$$\mathbf{F} \cdot \mathbf{n} = \frac{1}{a}(x^2 + y^2 + z^2) = \frac{a^2}{a} = a$$

Thus the integral is

$$\begin{aligned} & \int_S \mathbf{F} \cdot \mathbf{n} \, dS \\ = & \int_S a \, dS \\ = & \, a \int_S 1 \, dS \end{aligned}$$

The last integral is the surface area of the half-sphere of radius a, which is $2\pi a^2$. Thus the entire integral has value $2\pi a^3$.

6. The given triangle bounds the plane with equation $x + y + \frac{z}{2} = 1$, or $z = 2 - 2x - 2y$. This has parametrization $< x, y, 2 - 2x - 2y >$, where x and y are in the triangle bounded by $x = 0, y = 0, x + y = 1$. This parametrization gives $r_x \times r_y = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. We also need to calculate the curl of $\mathbf{F}$:

$$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} = (x-0)\mathbf{i} + (0-y)\mathbf{j} + (2x-1)\mathbf{k}$$

Thus, by Stokes' theorem,

$$\begin{aligned} & \int_C \mathbf{F} \cdot \, d\mathbf{r} \\ = & \int_S \text{curl}\mathbf{F} \cdot \, d\mathbf{r} \\ = & \int_D < x, -y, 2x - 1 > \cdot < 2, 2, 1 > \, dA \\ = & \int_0^1 \int_0^{1-x} 2x - 2y + 2x - 1 \, dy \, dx \\ = & \int_0^1 \int_0^{1-x} 4x - 1 - 2y \, dy \, dx \\ = & \int_0^1 \left((4x - 1)y - y^2\right)_0^{1-x} \, dx \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 (4x-1)(1-x) - (1-x)^2\,dx \\
&= \int_0^1 -5x^2 + 7x - 2\,dx \\
&= \frac{-5}{3} + \frac{7}{2} - 2 \\
&= \frac{-1}{6}
\end{aligned}
$$

7. By Stokes' theorem,

$$\int_S \text{curl}\mathbf{F} \cdot \mathbf{n}\,dS = \int_C \mathbf{F} \cdot\, dr$$

where C is the boundary of S. The given surface has its boundary curve the intersection of $z = 9 - x^2 - y^2$ and $z = 0$, so $x^2 + y^2 = 9$, in other words, the circle of radius 3 in the xy-plane. This has parametrization $(3\cos\theta, 3\sin\theta, 0)$. Thus

$$
\begin{aligned}
&\int_C \mathbf{F} \cdot\, dr \\
&= \int_C P\,dx + Q\,dy + R\,dz\ dt \\
&= \int_0^{2\pi} 3(0)(-3\sin\theta) + 4(3\cos\theta)(3\cos\theta) + 2(3\sin\theta)(0)\,d\theta \\
&= \int_0^{2\pi} 36\cos^2 d\,\theta \\
&= \int_0^{2\pi} 18 + 18\cos 2\theta\,d\theta \\
&= (18\theta + 9\sin 2\theta)_0^{2\pi} \\
&= 36\pi
\end{aligned}
$$

8. The flow is given by a surface integral. Since the surface bounds the cube of volume 1, so the cube $E = [0,1] \times [0,1] \times [0,1]$. Thus, by the divergence theorem, the flow is

$$
\begin{aligned}
&\int_S \mathbf{F} \cdot\, d\mathbf{S} \\
&= \int_E \text{div}\mathbf{F}\,dV
\end{aligned}
$$

$$
\begin{aligned}
&= \int_E z^2 + x \, dV \\
&= \int_0^1 \int_0^1 \int_0^1 z^2 + x \, dy \, dx \, dz \\
&= \int_0^1 \int_0^1 z^2 + x \, dx \, dz \\
&= \int_0^1 \left(z^2 x + \frac{x^2}{2} \right)_0^1 dz \\
&= \int_0^1 z^2 + \frac{1}{2} \, dz \\
&= \left(\frac{z^3}{3} + \frac{z}{2} \right)_0^1 \\
&= \frac{1}{3} + \frac{1}{2} \\
&= \frac{5}{6}
\end{aligned}
$$